Interference Alignment with Incomplete CSIT Sharing

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Abstract

This work[1] deals with Interference Alignment (IA) in a K-users MIMO Interference Channel with *incomplete* Channel State Information at the Transmitters (CSIT). Incompleteness of CSIT refers to the perfect knowledge at each TX of only a sub-matrix of the global channel matrix, where the sub-matrix is specific to each transmitter. This paper investigates the notion of IA feasibility for CSIT settings which are as incomplete as possible, as this can lead to feedback overhead reduction in practice. We distinguish between antenna configurations where (i) removing a single antenna makes IA unfeasible, referred to as *tightly-feasible* settings, and (ii) cases where extra antennas are available, referred to as *super-feasible* settings. We show the conditions for which IA is feasible in strictly incomplete CSIT scenarios, *even in tightly-feasible settings*. For such cases, we provide a CSIT allocation policy preserving IA feasibility while reducing significantly the amount of CSI required at the TXs. For super-feasible settings, we develop a heuristic CSIT allocation algorithm which exploits the additional antennas to further reduce the size of the CSIT allocation.

Index Terms

Interference alignment, Interference Channel, Channel State Information, Degrees-of-Freedom

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I. INTRODUCTION

Although multi-transmitter coordinated transmission such as Interference Alignment (IA) [2], [3] constitutes a promising tool to combat interference, coordination benefits go at the expense of acquiring accurate enough Channel State Information (CSI) at the Transmitters (TXs) and sharing it across all TXs whether explicitly or implicitly [4]. In the case of multi-antenna based IA without channel extension, which is the focus of this work, some form of CSI at the TXs (CSIT) is required to compute the precoders at each one of the TXs and can result in a significant overhead in practice.

The IA literature is rich in methods which have improved the efficiency of the precoding schemes at finite SNR and reduced the complexity of the algorithms [5]–[10]. Yet, the obtaining of the CSIT at the TXs represents a major obstacle to their practical use [11] and the study of how CSIT requirements can somehow be alleviated has become an active research topic in its own right [5], [6], [12]–[18]. Several approaches have been proposed in this direction and are briefly summarized below.

One strategy consists in developing iterative methods that can exploit local measurements made by the TXs on the reverse link or progressive feedback mechanisms [5], [6], [15]. Such methods rely on the fact that, through iterations, enough CSIT is acquired to allow convergence in a distributed manner toward a global IA solution. In [19], [20], the amount of information exchanged between the TXs is reduced by letting some TXs compute their precoder and share it instead of sharing the CSI. Yet, this is obtained at the cost of an increased delay because the improvement is obtained by letting the TXs successively compute their precoders. Also, this scheme is only applicable in some particular antenna configurations. In [21], [22], IA is adapted to the configurations of cellular networks. In [18], multi-user diversity is exploited to obtain approximately aligned interference without the requirement of full CSIT. In [10], the trade-off between serving jointly all the users via IA in a large IC or serving the users orthogonally in different frequency bands is investigated. To reduce the overhead due to the CSI feedback, an intermediate solution is found where the IC is split into smaller ICs to improve the overall efficiency of the transmission scheme.

Another line of work consists in studying the minimal CSI quantization bits (scaling with the SNR) that should be conveyed to the TXs to achieve the optimal number of degree of freedom (DoF) obtained under IA [13], [14].

Sharing the complete CSI to all the TXs becomes quickly challenging as the size of the IC increases. Let us consider for example the IC with K users where all the nodes have the same number of antennas given by $M = N = (K+1)/2$ so as to allow for IA feasibility with one stream per user. The multi-user channel, which should be then shared to every TX, contains a number of channel coefficients equal to $(K \cdot M) \cdot (K \cdot N) = K^2(K+1)^2/4$. Hence, the total feedback load scales as K^5 which represents an obstacle for the practical use of IA at even moderate values of K.

A common trait behind previous work dealing with limited CSIT for IA is that all TXs are typically assumed to have access to elements of the global CSI with the same representation quality. Yet, it will be shown that the fundamental shortcoming of such a framework is that it fails to properly reflect the actual CSIT requirements which in fact should depend on the antenna configuration.

Depending on the antenna configuration and the tightness level with which the IA feasibility conditions [23]–[27] are met, it is clear that not all channel vectors may need to be feedback to all TXs. This is fairly trivial in scenarios with extra antennas, which we denote as *super-feasible*, because these extra antennas can be used to zero-force (ZF) more sources of interference, hence lessening the need for actual alignment. However, it can be shown to be the case also in settings without extra antennas, which we denote as *tightly-feasible*. This shows the need to discuss IA feasibility, no longer in absolute terms but rather under the prism of CSIT assumptions.

To explore this problem, we introduce a novel CSIT framework whereby CSIT is no longer uniform across TXs. An IC with K users has *incomplete* CSIT when each of the K TXs acquires, through a given feedback and exchange mechanism left to be specified, a *subset* of the multi-user channel coefficients, with this subset being generally TX-dependent. In this framework, we define the *size* of a CSIT allocation as the total number of scalars forming the CSIT subsets known at the TXs.

Our main contributions are as follows.

- We formulate the problem of finding the CSIT allocation of minimal size which preserves IA feasibility. We show conditions under which IA is feasible with strictly incomplete CSIT.
- For *tightly-feasible* ICs, we present a CSIT allocation policy to the various TXs which preserves IA feasibility while reducing significantly the size of the CSIT feedback.
- For *super-feasible* ICs, we show the existence of a trade-off between the number of antennas and the sharing requirements. We provide a heuristic algorithm exploiting any given antenna number configuration to reduce further the size of the CSIT allocation.

Notations: We denote the Hadamard (or element-wise product) by the operator $\odot$ and by $\mathbb{1}_{i\in\mathcal{A}}$ the characteristic function which takes the value 1 if i belongs to the set $\mathcal{A}$, and zero otherwise. $\mathcal{N}(\mu,\sigma^2)$ denotes the complex circularly symmetric Gaussian distribution with mean μ and variance σ^2. The operator $\text{eig}_{\min}(\bullet)$ returns the eigenvector corresponding to the smallest eigenvalue of the matrix taken as argument and $|\mathcal{S}|$ is the the number of elements in the set $\mathcal{S}$. We write *"wlog"* for *"without loss of generality"*.

II. System Model

A. MIMO Interference Channel

We study the transmission in a K-user MIMO IC where all the RXs and the TXs are linked by a wireless channel. We consider a conventional channel model with the particularity of our model lying in the structure of the CSIT. We consider that each TX has its *own* CSIT in the form of a sub-matrix of the multi-user channel matrix. This specific information structure is referred to in this paper as *incomplete CSIT* and will be detailed in Subsection II-C. TX j is equipped with M_j antennas, RX i has N_i antennas, and TX j transmits a single stream to RX j. This IC is then denoted as $\prod_{k=1}^{K}(N_k,M_k)$. We consider exclusively single-stream transmissions and the extension to multiple streams will be discussed later in this work. When all the TXs and all the RXs have the same (resp. different) number of antennas (i.e., $(N,M)^K$), we say that the antenna configuration is *homogeneous* (resp. *heterogeneous*).

The channel from TX j to RX i is represented by the channel matrix $\mathbf{H}_{ij} \in \mathbb{C}^{N_i \times M_j}$ with its elements distributed according to a continuous probability distribution to ensure that all the channel matrices are almost surely full rank. The global multi-user channel matrix is denoted by $\mathbf{H} \in \mathbb{C}^{N_{\text{tot}} \times M_{\text{tot}}}$ where $N_{\text{tot}} \triangleq \sum_{k=1}^{K} N_k$ and $M_{\text{tot}} \triangleq \sum_{k=1}^{K} M_k$:

$$\mathbf{H} \triangleq \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \dots & \mathbf{H}_{1K} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \dots & \mathbf{H}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{K1} & \mathbf{H}_{K2} & \dots & \mathbf{H}_{KK} \end{bmatrix}. \tag{1}$$

TX i uses the TX beamformer $\boldsymbol{t}_i \in \mathbb{C}^{M_i \times 1}$ to transmit the data symbol s_i (i.i.d. $\mathcal{N}(0,1)$) to RX i. We consider the per-TX power constraint $\forall i \in \{1, \dots, K\}, \|\boldsymbol{t}\|_i^2 = P$. The received signal $\boldsymbol{y}_i \in \mathbb{C}^{N_i \times 1}$ at the i-th RX reads then as

$$\boldsymbol{y}_i = \mathbf{H}_{ii}\boldsymbol{t}_i s_i + \sum_{j=1, j \neq i}^{K} \mathbf{H}_{ij}\boldsymbol{t}_j s_j + \boldsymbol{\eta}_i \tag{2}$$

where $\boldsymbol{\eta}_i \in \mathbb{C}^{N_i \times 1}$ is the normalized noise at RX i and is i.i.d. $\mathcal{N}(0,1)$. The received signal $\boldsymbol{y}_i$ is then processed by a RX beamformer $\boldsymbol{g}_i^{\text{H}} \in \mathbb{C}^{1 \times N_i}$ to obtain an estimate of the data symbol s_i.

Our analysis deals with the achieveability of IA which means that the desired signal should be decoded free of interference at each RX. Equivalently, the RX beamformer $\boldsymbol{g}_i^{\text{H}}$ should zero-force (ZF) all the received interference which means fulfilling for all the interferers $j \neq i$

$$\boldsymbol{g}_i^{\text{H}} \mathbf{H}_{ij} \boldsymbol{t}_j = 0. \tag{3}$$

Thus, IA is feasible if the constraint (3) can be achieved at all the RXs for all the interfering streams. Note that this is equivalent to having the interference subspace at RX i span at most $N_i - 1$ dimensions.

B. *Feasibility Results*

1) Results from the Literature: We start by recalling some results from the literature on IA feasibility in a conventional IC with full CSIT sharing for the case of single stream transmission.

In [23], the notion of *proper* antenna configurations in introduced. An IC is said to be proper if and only if the number of variables in the RX and TX beamformers involved in any set of IA constraints is larger than the number of scalar equations. Following [23], let us denote by E_{ij} the IA equation (3) and by $\mathrm{var}(E_{ij})$ the set of free variables involved in this equation. It holds then

$$|\,\mathrm{var}(E_{ij})| = N_i - 1 + M_j - 1. \tag{4}$$

A system is proper if and only if

$$\forall \mathcal{I} \subseteq \mathcal{J}, |\mathcal{I}| \leq |\bigcup_{(i,j)\in\mathcal{I}} \mathrm{var}(E_{ij})| \tag{5}$$

where $\mathcal{J} \triangleq \{(i,j)|1 \leq i,j \leq K, i \neq j\}$ and $\mathcal{I}$ is an arbitrary subset of $\mathcal{J}$. In the homogeneous $(N,M)^K$ IC, this condition reads simply as $M + N \geq K + 1$. The following result has been later obtained in [26] and is restated here for convenience.

Theorem 1 ([26])**.** *IA is feasible in the* $\prod_{k=1}^{K}(N_k, M_k)$ *IC if and only if the antenna configuration is proper, i.e., if* (5) *is verified.*

2) Tightly-Feasible and Super-Feasible Settings: Whether the total number of variables is strictly larger than the number of equations will be shown to impact significantly the CSIT needed. Hence, we introduce the following definitions.

Definition 1. *An IC setting is called* tightly-feasible *if this IC is feasible and removing a single antenna at any TX or RX renders IA unfeasible. Equivalently, an IC is tightly-feasible if and only if it is* feasible *and*

$$\sum_{i=1}^{K} N_i + M_i = K(K+1). \tag{6}$$

The characterization follows directly from (5) applied with the set $\mathcal{I} = \mathcal{J}$.

Definition 2. *A feasible setting which does not verify the tightly-feasible condition is said to be*

super-feasible. *Equivalently, a super-feasible setting is a feasible setting such that*

$$\sum_{i=1}^{K} N_i + M_i > K(K+1). \tag{7}$$

3) New Formulation of the Feasibility Results: Condition (5) for properness requires verifying a number of conditions increasing exponentially with the size of the network. An interesting by-product of this work it to prove that condition (5) can be significantly simplified to obtain the following condition.

Theorem 2. *IA is feasible in the* $\prod_{k=1}^{K}(N_k, M_k)$ *IC if and only if*

$$\forall \mathcal{S}_{TX}, \mathcal{S}_{RX} \subseteq \mathcal{K}, \quad \mathcal{N}_{\mathrm{var}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}) \geq \mathcal{N}_{\mathrm{eq}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}) \tag{8}$$

where we have defined for arbitrary TX subset $\mathcal{S}_{TX}$ *and RX subset* $\mathcal{S}_{RX}$*:*

$$\mathcal{N}_{\mathrm{var}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}) \triangleq \sum_{i\in\mathcal{S}_{RX}} N_i - 1 + \sum_{i\in\mathcal{S}_{TX}} M_i - 1, \qquad \mathcal{N}_{\mathrm{eq}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}) \triangleq \sum_{k\in\mathcal{S}_{TX}} \sum_{j\in\mathcal{S}_{RX}, j\neq k} 1 \tag{9}$$

as respectively the number of variables and the number of equations stemming from the subset of RXs $\mathcal{S}_{RX}$ *and the subset of TXs* $\mathcal{S}_{TX}$*.*

Proof: A detailed proof is given in Appendix B. ■

The criterion (8) can be verified in polynomial time and provides in addition more insights into IA feasibility: The feasibility of IA in the full setting is verified by analyzing the feasibility of IA in all the *sub-ICs* included in the full IC.

Note that the setting obtained after selection of the RXs inside $\mathcal{S}_{\mathrm{RX}}$ and the TXs inside $\mathcal{S}_{\mathrm{TX}}$ is not a conventional IC due to the fact that the TXs and the RXs are not necessarily *paired*. To model this scenario, we introduce the notion of *generalized IC* which is shown to play an important role in understanding the relation between IA feasibility and CSIT.

4) Generalized Interference Channels: We refer to an IC in which at least one TX or RX does not have its paired RX or TX included in the IC as a *generalized IC*. We represent this by writing a "*" instead of the number of antennas of the paired RX or TX. For example, an

IC containing TXs 1, 2, and 3 but only RXs 1, 2, and 4 (with all the TXs and the RXs having two antennas) is denoted by $(2,2).(2,2).(*,2).(2,*)$. The IA feasibility criterion (8) is trivially extended to generalized ICs by verifying the condition for all the sets of TXs and RXs included in the generalized IC.

C. Incomplete CSIT Model

The feasibility results from the literature, which we have recalled above, have always made use of an implicit full CSIT assumption. Hence, surprisingly, the problem of revisiting the feasibility conditions under the light of a partial CSIT sharing framework has not been addressed before. To fill this gap, it is necessary to introduce a new model to take into account the partial CSIT sharing capability of the TXs.

A TX has either perfect knowledge of a channel coefficient or no information at all on that element. We represent the CSIT structure at TX j by the *CSIT matrix* $\mathbf{A}^{(j)} \in \{0,1\}^{N_{\text{tot}} \times M_{\text{tot}}}$ such that $\{\mathbf{A}^{(j)}\}_{ik} = 1$ if $\{\mathbf{H}\}_{ik}$ is known at TX j, and 0 otherwise. Denoting by $\mathbf{H}^{(j)}$ the available CSI at TX j, we obtain

$$\mathbf{H}^{(j)} = \mathbf{A}^{(j)} \odot \mathbf{H}. \tag{10}$$

We define the CSIT allocation $\mathcal{A}$ as the set of CSI representations available at all TXs:

$$\mathcal{A} = \{\mathbf{A}^{(j)} | \mathbf{A}^{(j)} \in \{0,1\}^{N_{\text{tot}} \times M_{\text{tot}}}, j = 1, \ldots, K\} \tag{11}$$

and we define the space $\mathbb{A}$ containing all the possible CSIT allocations. We can then define the *size* of an incomplete CSIT allocation as follows.

Definition 3. *The size of a CSIT allocation $\mathcal{A}$, denoted by $s(\mathcal{A})$, is equal to the overall number of complex channel coefficients feedback to the TXs. Thus,*

$$s(\mathcal{A}) \triangleq \sum_{j=1}^{K} \|\mathbf{A}^{(j)}\|_{\text{F}}^2. \tag{12}$$

To check whether IA feasibility is preserved with a given CSIT allocation, we introduce the function f_{Feas} which takes as argument a CSIT allocation $\mathcal{A}$ and an antenna configura-

tion $\prod_{k=1}^{K}(N_k, M_k)$ and returns 1 if IA is feasible with these parameters and 0 otherwise. Note that this means that there exists one algorithm achieving IA with this CSIT allocation but it does not precise the algorithm. We also define the set $\mathbb{A}_{\text{Feas}}$ containing all the CSIT allocations for which IA is feasible. Hence,

$$\mathbb{A}_{\text{Feas}} \triangleq \{\mathcal{A}|\mathcal{A} \in \mathbb{A}, f_{\text{Feas}}(\mathcal{A}, \prod_{k=1}^{K}(N_k, M_k)) = 1\}. \tag{13}$$

Only the interfering channel matrices $\mathbf{H}_{ij}$ with $i \neq j$ are required to fulfill the IA constraints, and not the direct channel matrices $\mathbf{H}_{jj}$. Thus, from a DoF point of view, we can always skip the direct channel matrices $\mathbf{H}_{jj}$ in the feedback, which leads to the following definition.

Definition 4. *A* complete *CSIT allocation, denoted by $\mathcal{A}_{\text{comp}}$, is defined by the knowledge of all the interfering channel matrices $\mathbf{H}_{ij}$ with $i \neq j$ but not the direct channel matrices $\mathbf{H}_{jj}$. Thus, the size of a complete CSIT allocation is*

$$s\left(\mathcal{A}_{\text{comp}}\right) = K\left(N_{tot}M_{tot} - \sum_{i=1}^{K} N_i M_i\right). \tag{14}$$

A CSIT allocation with a size smaller than $s(\mathcal{A}_{\text{comp}})$ is said to be strictly incomplete.

At this stage, a natural question is to find the most incomplete CSIT allocation which preserves the feasibility of IA, i.e.,

$$\mathcal{A}_{\min} = \underset{\mathcal{A} \in \mathbb{A}_{\text{Feas}}}{\operatorname{argmin}}\, \mathrm{s}(\mathcal{A}). \tag{15}$$

Note that we can limit our study to the IA feasible settings where $\mathcal{A}_{\text{comp}} \in \mathbb{A}_{\text{Feas}}$.

III. IA WITH INCOMPLETE CSIT FOR TIGHTLY-FEASIBLE CHANNELS

A. General Criterion

1) Parametrization of the CSIT Allocation: In order to write concisely our results, we need to introduce a last notation. With simple words, the matrix $\mathbf{A}_{\mathcal{S}^{\text{RX}},\mathcal{S}^{\text{TX}}}$, where $\mathcal{S}^{\text{RX}}$ is a set of RXs and $\mathcal{S}^{\text{TX}}$ a set of TXs, is defined such that $\mathbf{A}_{\mathcal{S}^{\text{RX}},\mathcal{S}^{\text{TX}}} \odot \mathbf{H}$ contains all the channel coefficients relative to the generalized sub-IC formed by the sets of RXs $\mathcal{S}^{\text{RX}}$ and the set of TXs $\mathcal{S}^{\text{TX}}$, at

the exception of the direct channel matrices $\mathbf{H}_{jj}$.

This means that the matrix $\mathbf{A}_{\mathcal{S}^{\text{RX}},\mathcal{S}^{\text{TX}}}$ of size $N_{\text{tot}} \times M_{\text{tot}}$ has its only nonzero elements chosen to verify

$$\forall x \neq y, x \in \mathcal{S}^{\text{RX}}, y \in \mathcal{S}^{\text{TX}}, \quad \left(\mathbf{E}_x^{\text{RX}}\right)^{\text{T}} \mathbf{A}_{\mathcal{S}^{\text{RX}},\mathcal{S}^{\text{TX}}} \mathbf{E}_y^{\text{TX}} = \left(\mathbf{E}_x^{\text{RX}}\right)^{\text{T}} \mathbf{1}_{N_{\text{tot}} \times M_{\text{tot}}} \mathbf{E}_y^{\text{TX}} \tag{16}$$

with $\mathbf{E}_n^{\text{TX}} \triangleq \left[\mathbf{0}_{\sum_{k=1}^{n-1} M_k \times M_n}, \mathbf{I}_{M_n}, \mathbf{0}_{\sum_{k=n+1}^{K} M_k \times M_n}\right]^{\text{T}}$ and the matrix $\mathbf{E}_n^{\text{RX}}$ defined similarly, solely with N_i replacing M_i.

2) Main Theorem: We can now state one of our main results.

Theorem 3. *In a tightly-feasible $\prod_{k=1}^{K}(N_k, M_k)$ IC, let us assume that there exists a tightly-feasible generalized sub-IC formed by the TX-set $\mathcal{S}_{TX}$ and the RX-set $\mathcal{S}_{RX}$, which means that*

$$\mathcal{N}_{\text{var}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}) = \mathcal{N}_{\text{eq}}(\mathcal{S}_{RX}, \mathcal{S}_{TX}). \tag{17}$$

Then the incomplete CSIT allocation $\mathcal{A} = \{\mathbf{A}^{(j)} | j = 1, \ldots, K\}$ preserves IA feasibility, i.e., $\mathcal{A} \in \mathbb{A}_{\text{Feas}}$ if

$$\forall j \in \mathcal{S}_{TX}, \mathbf{A}^{(j)} = \mathbf{A}_{\mathcal{S}_{RX},\mathcal{S}_{TX}}, \qquad \forall j \notin \mathcal{S}_{TX}, \mathbf{A}^{(j)} = \mathbf{A}_{\mathcal{K},\mathcal{K}} = \mathbf{1}_{N_{tot} \times M_{tot}}. \tag{18}$$

Proof: A detailed proof is provided in Appendix C. ■

Hence, the existence of a tightly-feasible generlized sub-IC strictly included in the full IC yields the existence of a strictly incomplete CSIT allocation preserving IA feasibility. From the iterative use of this property, we will show in the following that a reduced CSIT allocation ensuring IA feasibility is obtained if *each TX just receives the CSIT relative to the smallest tightly-feasible IC to which it belongs*.

Applying Theorem 3 in an homogeneous setting gives a more pessimistic result.

Corollary 1. *In the tightly-feasible setting $(N, M)^K$ (i.e., $M + N = K + 1$) with $M \neq 1$ and $M \neq K$, there exists no generalized tightly-feasible sub-IC which is strictly included in the*

full IC. Hence, the previous sufficient condition leads to no CSIT reduction.

Proof: The proof follows easily by evaluating (17) in an homogeneous setting and is omitted for brevity. ■

If the full IC is tightly-feasible, a strictly smaller IC can be tightly-feasible only by exploiting the heterogeneity in the antenna configuration. Hence, the sufficient condition given in Theorem 3 cannot be fulfilled in any tightly-feasible homogeneous IC with $M \neq 1$ and $M \neq K$.

B. CSIT Allocation Algorithm

We now describe an incomplete CSIT allocation policy based on Theorem 3. Interestingly, exhibiting an incomplete CSIT allocation that preserves IA feasibility does not say how IA shall be performed on the basis of this CSIT allocation. The corresponding problem of designing an algorithm achieving IA based on the incomplete CSIT allocation is tackled in Subsection III-C.

The CSIT allocation algorithm takes as input the antenna configuration $\prod_{k=1}^{K}(N_k, M_k)$ and returns as output the incomplete CSIT allocation $\mathcal{A} = \{\mathbf{A}^{(j)} | j = 1, \ldots, K\}$ such that

$$\forall j, \mathbf{A}^{(j)} = \mathbf{A}_{\mathcal{S}_{\text{RX}}^{(j)}, \mathcal{S}_{\text{TX}}^{(j)}}. \tag{19}$$

Let us consider wlog the problem of allocating the CSI to TX j.

Initialization: We first define an initial pair of sets $\mathcal{S} \triangleq (\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}})$ initialized such that

$$\mathcal{S} = (\emptyset, \{j\}). \tag{20}$$

The remaining TXs (without considering TX j) are ordered by increasing number of antennas, i.e., with the permutation σ_{TX} verifying

$$\forall i = \{1, \ldots, K-2\}, M_{\sigma_{\text{TX}}(i)} \leq M_{\sigma_{\text{TX}}(i+1)}. \tag{21}$$

and symmetrically, the RXs are ordered by increasing number of antennas, i.e., with the permutation σ_{RX} verifying

$$\forall i = \{1, \ldots, K-1\}, N_{\sigma_{\text{RX}}(i)} \leq N_{\sigma_{\text{RX}}(i+1)}. \tag{22}$$

In case of equality, we order the TXs to ensure that

$$(M_{\sigma_{\text{TX}}(i)} = M_{\sigma_{\text{TX}}(i+1)}) \Rightarrow N_{\sigma_{\text{TX}}(i)} \geq N_{\sigma_{\text{TX}}(i+1)}. \tag{23}$$

Similarly, the RX ordering is modified to ensure that

$$(N_{\sigma_{\text{RX}}(i)} = N_{\sigma_{\text{RX}}(i+1)}) \Rightarrow M_{\sigma_{\text{RX}}(i)} \geq M_{\sigma_{\text{RX}}(i+1)}. \tag{24}$$

This ensures that unpaired TXs and RXs are selected first in case of equality in the number of antennas if the TXs and the RXs are selected respectively following the orderings σ_{TX} and σ_{RX}.

Update at step n: Let us assume that we are given the pair of sets $\mathcal{S} = (\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}})$.

1) If equation (17) is verified with the sets $\mathcal{S}_{\text{RX}}$ and $\mathcal{S}_{\text{TX}}$, the algorithm has reached its end. We set $\mathcal{S}_{\text{RX}}^{(j)} = \mathcal{S}_{\text{RX}}$, $\mathcal{S}_{\text{TX}}^{(j)} = \mathcal{S}_{\text{TX}}$ and

$$\mathbf{A}^{(j)} = \mathbf{A}_{\mathcal{S}_{\text{RX}}^{(j)}, \mathcal{S}_{\text{TX}}^{(j)}}. \tag{25}$$

2) If equation (17) does not hold, we verify whether adding the next RX adds more equations than variables, i.e.,

$$\begin{aligned}\mathcal{N}_{\text{var}}(\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}}) - \mathcal{N}_{\text{eq}}(\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}}) \geq \mathcal{N}_{\text{var}}(\{\mathcal{S}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}_{\text{RX}}|+1)\}, \mathcal{S}_{\text{TX}}) \\ - \mathcal{N}_{\text{eq}}(\{\mathcal{S}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}_{\text{RX}}|+1)\}, \mathcal{S}_{\text{TX}})\end{aligned} \tag{26}$$

- If (26) is verified, we set

$$\mathcal{S}^{\text{RX}} = \{\mathcal{S}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}_{\text{RX}}|+1)\} \tag{27}$$

  and we start over at step $n+1$.

- If (26) is not verified, then
  - If $|\mathcal{S}_{\text{TX}}| < K$, we increase the set of TXs as

$$\mathcal{S}_{\text{TX}} = \{\mathcal{S}_{\text{TX}}, \sigma_{\text{TX}}(|\mathcal{S}_{\text{TX}}|+1)\} \tag{28}$$

    and we start over at step $n+1$.

– If $|\mathcal{S}_{\text{TX}}| = K$, then the algorithm has reached its end and we set $\mathcal{S}_{\text{RX}}^{(j)} = \mathcal{S}_{\text{RX}}$ and $\mathcal{S}_{\text{TX}}^{(j)} = \mathcal{S}_{\text{TX}}$ and

$$\mathbf{A}^{(j)} = \mathbf{A}_{\mathcal{S}_{\text{RX}}^{(j)}, \mathcal{S}_{\text{TX}}^{(j)}}. \tag{29}$$

C. IA Algorithm for Incomplete CSIT Allocation

We consider now the CSIT allocation $\mathcal{A}$ to be given and we describe a novel IA algorithm which achieves IA using the incomplete CSIT allocation. The algorithm runs in a distributed fashion at each TX. This IA algorithm, which we denote by f_{IA}, takes as input the antenna configuration, the CSIT allocation policy, and the channel coefficients known at the TX, and returns the beamformer at TX j. Thus, we can write at TX j

$$\boldsymbol{t}_j = f_{\text{IA}}(\prod_{k=1}^{K}(N_k, M_k), \mathcal{A}, \mathbf{H}^{(j)}). \tag{30}$$

1) IA Algorithm for Effective Channels: We start by introducing an IA algorithm f_{Eff} which will be a building block for our algorithm. It consists in running an IA algorithm over the *effective* channel obtained once a fraction of the TX beamformers have been fixed. Hence, taking as input the set containing the fixed beamformers $\mathcal{B}_{\text{TX}}^{\text{Fix}}$ and a channel matrix $\mathbf{G}$, it returns as output the set of beamformers $\mathcal{B}^{\text{TX}}$ obtained after having run a conventional IA algorithm from the literature over this effective channel. Note that since the TX beamformers inside $\mathcal{B}_{\text{TX}}^{\text{Fix}}$ are not modified, it holds that $\mathcal{B}_{\text{TX}}^{\text{Fix}} \subset \mathcal{B}_{\text{TX}}$. We can then write

$$\mathcal{B}_{\text{TX}} = f_{\text{Eff}}(\mathbf{G}, \mathcal{B}_{\text{TX}}^{\text{Fix}}). \tag{31}$$

A number of IA algorithms can be run over the effective channel, and we will use the most simple IA algorithm called the *min-leakage* algorithm [5]. We recall for completeness its mains steps in Appendix A. Our IA algorithm is obtained from the min-leakage algorithm after two simple modifications of the update formulas [Cf. equations (46) and (47)]:

- The update of the beamformers is done by considering all the interfering links and not by summing from 1 to K because we consider here *generalized* ICs.
- The TX beamformers contained in $\mathcal{B}_{\text{TX}}^{\text{Fix}}$ are kept unchanged.

2) Precoding with the Incomplete CSIT: Let us consider now the precoding at TX j with the CSIT allocation $\mathbf{H}^{(j)} = \mathbf{A}_{\mathcal{S}_{\text{RX}}^{(j)},\mathcal{S}_{\text{TX}}^{(j)}} \odot \mathbf{H}$.

The first step is to verify whether there is a TX k having its CSIT included in the CSIT at TX j, i.e., whether there exists a $k \neq j$ such that

$$\mathcal{S}_{\text{RX}}^{(k)} \subseteq \mathcal{S}_{\text{RX}}^{(j)}, \quad \mathcal{S}_{\text{TX}}^{(k)} \subseteq \mathcal{S}_{\text{TX}}^{(j)}. \tag{32}$$

If this is the case, TX j computes first the beamformer of TX k before computing its own beamformer. This process continues recursively until a TX is obtained for which no smaller CSIT allocation verifying (32) can be found. Hence, TX j computes first the TX beamformers corresponding to the TXs with the smallest CSIT allocation verifying (32) as follows:

$$\mathcal{B}_{\text{TX}}^{k_0} = f_{\text{Eff}}(\mathbf{H}^{\tilde{(k_0)}}, \emptyset) \tag{33}$$

where $\mathbf{H}^{(k_0)}$ represents the smallest CSIT allocation obtained and $\mathbf{H}^{\tilde{(k_0)}}$ is the matrix of reduced size obtained after removing all-zero rows and columns. Once this is done, the beamformers corresponding to the TXs with the next smallest CSIT allocation verifying (32) are computed. This process is repeated further such that at each step i

$$\mathcal{B}_{\text{TX}}^{k_i} = f_{\text{Eff}}(\mathbf{H}^{\tilde{(k_i)}}, \mathcal{B}_{\text{TX}}^{k_{i-1}}). \tag{34}$$

The last step corresponds to the CSIT allocation of TX j which has thereby computed its beamformer[2].

3) Achieveability of Interference Alignment: We have described a precoding algorithm but it remains to prove that IA is indeed achieved.

Theorem 4. *The CSIT allocation policy $\mathcal{A}$ obtained with the incomplete CSIT allocation algorithm preserves IA feasibility: $\mathcal{A} \in \mathbb{A}_{\text{Feas}}$. Furthermore, the CSIT allocation $\mathcal{A}$ allocates to TX j*

[2]Note that it might happen that all CSIT allocations are not included in each other but form instead different *chains* of CSIT included in the CSIT of TX j. Hence, the previous notation is not rigorous. Yet, the same process of iteratively computing the beamformers is carried out for each chain such that this does not lead to any difficulty. This detail has been omitted for the sake of clarity.

the CSI relative to the smallest *tightly-feasible sub-IC to which it belongs.*

Proof: A detailed proof is provided in Appendix D. ∎

Remark: The TXs in $\mathcal{S}_{\text{TX}}^{(j)}$ and the RXs in $\mathcal{S}_{\text{RX}}^{(j)}$, as returned by the CSIT allocation algorithm, form together the smallest tightly-feasible setting containing TX j. If the algorithm is initialized with $\mathcal{S}_{\text{TX}}^{(j)} = \emptyset$ instead of $\mathcal{S}_{\text{TX}}^{(j)} = \{j\}$, the smallest tightly-feasible generalized IC is obtained (without the constraint of containing TX j). Hence, this algorithm can also be used to verify the IA feasibility of an antenna configuration.

We will now discuss an example illustrating the operational meaning of our approach.

D. Example of Tightly-Feasible

We consider the IC formed by the antenna configuration $(2,3).(2,4).(3,5).(3,2).(4,2)$. The algorithm presented in Subsection III-B returns the CSIT allocation

$$\begin{aligned}
&\mathcal{A} = \{\mathbf{A}^{(1)} = \mathbf{A}_{\{1,2,3\},\{4,5,1\}}, \mathbf{A}^{(2)} = \mathbf{A}_{\{1,2,3,4\},\{1,2,4,5\}}, \mathbf{A}^{(3)} = \mathbf{A}_{\{1,2,3,4,5\},\{1,2,3,4,5\}},\\
&\mathbf{A}^{(4)} = \mathbf{A}_{\{1,2\},\{4,5\}}, \mathbf{A}^{(5)} = \mathbf{A}_{\{1,2\},\{4,5\}}\}
\end{aligned} \tag{35}$$

which indicates for example that TX 4 receives the CSI relative to the *generalized* IC formed by TX 4 and 5 and RX 1 and 2. The size of the incomplete CSIT allocation obtained is equal to 346 while the complete CSIT allocation has a size of 900.

TX 4 and TX 5 have only the CSI required to align their interference at RX 1 and RX 2, which is the first step of the IA algorithm. Once this is done, TX 1 can then design its beamformer to align its interference on the interference subspace created by TX 4 and TX 5 at RX 2 and RX 3. Proceeding further, TX 2 align its interference on the interference subspace spanned at RX 1, RX 3, and RX 4 by the previous TX beamformers. At this step, all the interference subspaces are already spanned so that TX 3 can use its 5 antennas to align its interference at all the RXs.

As described in Subsection III-B, each TX computes the beamformers of the TXs having a CSIT allocation included in its own CSI before computing its own TX beamformer. For example, all the TXs start here by computing the beamformers of TX 4 and TX 5.

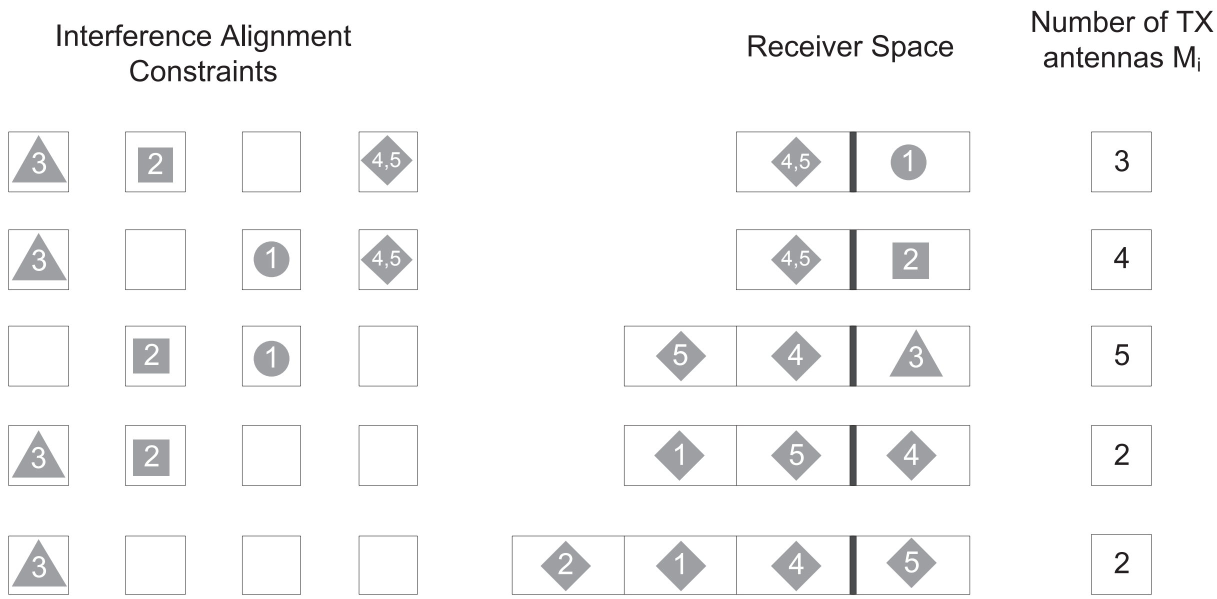

Fig. 1. Symbolic Representation of the IA algorithm with incomplete CSIT for the tightly-feasible IC $(2,3).(2,4).(3,5).(3,2).(4,2)$.

To verify the achievement of IA, we introduce a symbolic representation of IA in Fig. 1. We represent the dimensions available at RX i by an array of N_i boxes. The first box on the right represents the dimension taken by the signal while the other boxes represent the dimensions left free for the interference. For each RX, another box indicates if a TX precodes its signal so as to align with the interference subspace, thus creating no additional dimension of interference. If this is not the case, the stream transmitted by this TX creates a dimension of interference at the RX considered.

In this symbolic representation, a precoding scheme achieves IA if and only if the number of interfering dimensions at a RX does not exceed the number of boxes (dimensions) available at the RX while ensuring that each TX fulfills a number of IA constraint attainable with its antenna configurations (i.e. at most $M-1$ IA constraints if this TX has M antennas). When the TX beamformers are not obtained by ZF but via the alternative iteration of the min-leakage algorithm, we represent this by writing the indices of the interferers both in the IA box and in the RX box.

We can observe in Fig. 1 that for all j, TX j align its interference at M_j-1 RXs, and for all i, the interference subspace at RX i spans N_i-1 dimensions. One can also notice that the setting is indeed tightly-feasible since removing an antenna at any TX or RX makes IA unfeasible.

The intuition behind the IA algorithm for incomplete CSIT is to break the IA into successive steps. The successive precoding steps are symbolized by the different columns on the left of the symbolic representation.

IV. Interference Alignment with Incomplete CSIT for Super-Feasible Channels

The previous section indicates how CSIT savings can be obtained for tightly feasible scenarios. When additional antennas are available, the intuition goes that further CSIT savings should be possible at no cost in terms of IA feasibility. We now investigate this question.

A distinct feature of super-feasible settings is that there must exist a corresponding tightly-feasible setting that can be obtained by keeping all TXs and RXs and simply ignoring certain antennas among the overall antenna set. Clearly, there are generally multiple ways for arriving at a tightly-feasible setting from a super-feasible one. Depending on the choice of which antennas are ignored in the initial super-feasible setting, the obtained tightly-feasible will exhibit particular CSIT requirements.

As a consequence, we consider the following optimization problem instead of considering directly (10):

$$\begin{aligned} \mathcal{A} = \operatorname*{argmin}_{\mathcal{A}\in\mathbb{A}} \min_{\prod_{k=1}^{K}(N'_k,M'_k)} \mathrm{s}(\mathcal{A}) \quad & \text{s.t. } f_{\mathrm{Feas}}(\mathcal{A},\prod_{k=1}^{K}(N'_k,M'_k)) = 1 \\ & \text{s.t. } \sum_{i=1}^{K} M'_i + N'_i - (K+1)K = 0 \\ & \text{s.t. } 1 \leq M'_i \leq M_i \text{ and } 1 \leq N'_i \leq N_i. \end{aligned} \tag{36}$$

Hence, the problem of finding the minimal CSIT allocation has been reduced to finding the tightly-feasible setting (containing all the users) included in the full super-feasible setting which requires the smallest CSIT allocation. Since a CSIT allocation algorithm has been derived for

tightly-feasible settings, it remains only to determine which RXs or TXs should not fully exploit their antennas to ZF interference dimensions, i.e., where some antennas should be ”removed” in terms of IA feasibility.

Practically, the antennas will not be ”removed” but simply not exploited in the IA process and used instead to increase the received power, for example. Specifically, let us write the singular value decomposition of $\mathbf{H}_{ii}$ is $\mathbf{H}_{ii} = \mathbf{U}_i \mathbf{\Sigma}_i \mathbf{V}_i^{\mathrm{H}}$ with $\mathbf{V}_i = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{M_i}] \in \mathbb{C}^{M_i \times M_i}$ and $\mathbf{U}_i = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{N_i}] \in \mathbb{C}^{N_i \times N_i}$ two unitary matrices and $\mathbf{\Sigma}_i = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\min(M_i,N_i)}, 0, \ldots, 0)$.

Setting the TX beamformer $\boldsymbol{t}_i$ as $\boldsymbol{t}_i = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{M_i-1}]\boldsymbol{t}_i'$ with $\boldsymbol{t}_i' \in \mathbb{C}^{M_i-1\times 1}$ is equivalent in terms of IA feasibility to removing one antenna at TX i such that we will in the following simply say that the antennas are ”removed”[3].

This problem is combinatorial in the total number of TXs and RXs such that exhaustive search through all the possible solutions is only practical for small settings. As a consequence, we provide in the following a CSIT allocation policy which exploits heuristically the additional antennas available to reduce the size of the CSIT allocation. The heuristic behind the algorithm comes from the insight gained in the analysis of tightly-feasible settings. This insight suggests that the more heterogeneous is the antenna configuration, the smaller is the size of the CSIT allocation.

A. CSIT allocation Algorithm

We consider in the following an heterogeneous IC and we denote by S the total number of additional antennas in the sense that S is defined as

$$S \triangleq \sum_{i=1}^{K} M_i + N_i - (K+1)K. \tag{37}$$

The following algorithm will provide the pair of sets $\mathcal{S}^{\mathrm{NT}} = (\mathcal{S}^{\mathrm{NT}}_{\mathrm{RX}}, \mathcal{S}^{\mathrm{NT}}_{\mathrm{TX}})$ containing respectively the RXs and the TXs where the additional antennas should be ”removed”. Once these antennas have been ”removed”, the incomplete CSIT allocation policy for tightly-feasible can be

[3]Note that this step can be applied similarly on the RX side and that this process on the TX side requires the CSI relative to the direct channel.

applied to obtain the incomplete CSIT allocation. Note that we need to ensure that the antennas are removed at the right nodes such that IA feasibility is preserved.

Initialization: We define inside the algorithm a *virtual antenna configuration* $\prod_{i=1}^{K}(N_i^{\text{v}}, M_i^{\text{v}})$ which we initialize with the true antenna configuration $N_i^{\text{v}} = N_i, M_i^{\text{v}} = M_i$. We then define the two sets that will be given as output $\mathcal{S}_{\text{RX}}^{\text{NT}} = \emptyset, \mathcal{S}_{\text{TX}}^{\text{NT}} = \emptyset$.

Step n: We start by ordering the TXs inside $\prod_{i=1}^{K}(N_i^{\text{v}}, M_i^{\text{v}})$ by increasing number of antennas with the permutation σ_{TX}: $\forall i, M_{\sigma_{\text{TX}}(i)}^{\text{v}} \leq M_{\sigma_{\text{TX}}(i+1)}^{\text{v}}$.

Similarly, the RXs are ordered by increasing number of antennas[4], i.e., with the permutation σ_{RX} such that $\forall i, N_{\sigma_{\text{RX}}(i)}^{\text{v}} \leq N_{\sigma_{\text{RX}}(i+1)}^{\text{v}}$.

In case of equality, the permutation is modified such that if $M_{\sigma_{\text{TX}}(i)}^{\text{v}} = M_{\sigma_{\text{TX}}(i+1)}^{\text{v}}$, then

- If both RXs $\sigma_{\text{TX}}(i)$ and $\sigma_{\text{TX}}(i+1)$ belong to the IC, then $N_{\sigma_{\text{TX}}(i)}^{\text{v}} \geq N_{\sigma_{\text{TX}}(i+1)}^{\text{v}}$.
- If one of the two RXs does not belong to the IC, then it is RX $\sigma_{\text{TX}}(i)$, i.e, $N_{\sigma_{\text{TX}}(i)}^{\text{v}} = *$.

We apply the same process symmetrically for the permutation σ_{RX}. Similar to the algorithm for tightly-feasible ICs, this modification ensures that non-paired TXs and RXs are chosen first when several TXs or RXs have the same number of antennas. We define also by respectively K_{RX}^{v} and K_{TX}^{v} the number of RXs and the number of TXs actually in the generalized IC $\prod_{i=1}^{K}(N_i^{\text{v}}, M_i^{\text{v}})$.

1) a) We now define a set $\mathcal{S}^{\text{v}} \triangleq (\mathcal{S}_{\text{RX}}^{\text{v}}, \mathcal{S}_{\text{TX}}^{\text{v}})$. If $K_{\text{RX}}^{\text{v}} > 0$, we start by setting $\mathcal{S}^{\text{v}} = (\{\sigma_{\text{RX}}(1)\}, \emptyset)$. If (17) is not fulfilled with this choice of $\mathcal{S}^{\text{v}}$ and $K_{\text{TX}}^{\text{v}} > 0$, we reinitialize it with $\mathcal{S}^{\text{v}} = (\emptyset, \{\sigma_{\text{TX}}(1)\})$.

   b) If the equality is reached in (17) with $\mathcal{S}^{\text{v}}$, the sub-IC obtained is tightly-feasible and we update the antenna configuration to the one of the effective IC once IA is

[4]The RXs (resp. the TXs) outside the IC (with a "*" as number of antennas) are not taken into account in the RX ordering (resp. TX ordering).

fulfilled in this sub-IC:

$$\forall i \in \mathcal{S}^{\text{v}}_{\text{RX}}, N^{\text{v}}_i = *, \qquad \forall i \in \mathcal{S}^{\text{v}}_{\text{TX}}, M^{\text{v}}_i = *.$$
$$\forall i \notin \mathcal{S}^{\text{v}}_{\text{RX}}, \begin{cases} N^{\text{v}}_i = N^{\text{v}}_i - |\mathcal{S}^{\text{v}}_{\text{TX}}|, & \text{if } i \notin \mathcal{S}^{\text{v}}_{\text{TX}}. \\ N^{\text{v}}_i = N^{\text{v}}_i - (|\mathcal{S}^{\text{v}}_{\text{TX}}| - 1) & \text{if } i \in \mathcal{S}^{\text{v}}_{\text{TX}}. \end{cases} \tag{38}$$
$$\forall i \notin \mathcal{S}^{\text{TX}}_{\text{v}}, \begin{cases} M^{\text{v}}_i = M^{\text{v}}_i - |\mathcal{S}^{\text{v}}_{\text{RX}}|, & \text{if } i \notin \mathcal{S}^{\text{v}}_{\text{RX}}. \\ M^{\text{v}}_i = M^{\text{v}}_i - (|\mathcal{S}^{\text{v}}_{\text{RX}}| - 1) & \text{if } i \in \mathcal{S}^{\text{v}}_{\text{RX}}. \end{cases}$$

We then start over at step $n+1$.

c) If equation (17) is not verified.

- If $|\mathcal{S}^{\text{v}}_{\text{RX}}| < K^{\text{v}}_{\text{RX}}$, we verify whether adding the next RX adds more equations than variables, i.e.,

$$\mathcal{N}_{\text{var}}(\mathcal{S}^{\text{v}}_{\text{RX}}, \mathcal{S}^{\text{v}}_{\text{TX}}) - \mathcal{N}_{\text{eq}}(\mathcal{S}^{\text{v}}_{\text{RX}}, \mathcal{S}^{\text{v}}_{\text{TX}}) \geq \mathcal{N}_{\text{var}}(\{\mathcal{S}^{\text{v}}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}^{\text{v}}_{\text{RX}}| + 1)\}, \mathcal{S}^{\text{v}}_{\text{TX}}) - \mathcal{N}_{\text{eq}}(\{\mathcal{S}^{\text{v}}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}^{\text{v}}_{\text{RX}}| + 1)\}, \mathcal{S}^{\text{v}}_{\text{TX}}) \tag{39}$$

  If (39) is verified, we set

$$\mathcal{S}^{\text{v}}_{\text{RX}} = \{\mathcal{S}^{\text{v}}_{\text{RX}}, \sigma_{\text{RX}}(|\mathcal{S}^{\text{v}}_{\text{RX}}| + 1)\} \tag{40}$$

  and we start over at step $1.b)$.

- If $|\mathcal{S}^{\text{v}}_{\text{TX}}| < K^{\text{v}}_{\text{TX}}$, we increase the set of TXs as

$$\mathcal{S}^{\text{v}}_{\text{TX}} = \{\mathcal{S}^{\text{v}}_{\text{TX}}, \sigma_{\text{TX}}(|\mathcal{S}^{\text{v}}_{\text{TX}}| + 1)\} \tag{41}$$

  and we start over at step $1.b)$.

2) Otherwise, there is no tightly-feasible sub-IC and *removing an antenna cannot render IA unfeasible*.

- If $K^{\text{v}}_{\text{TX}} > 0$, we compute

$$M^{\text{v}}_{\sigma_{\text{TX}}(1)} = M^{\text{v}}_{\sigma_{\text{TX}}(1)} - 1, \quad \mathcal{S}^{\text{NT}}_{\text{TX}} = \{\mathcal{S}^{\text{NT}}_{\text{TX}}, \sigma_{\text{TX}}(1)\} \tag{42}$$

and we start at step $n+1$.

- If $K^{\mathrm{v}}_{\mathrm{TX}}=0$ but $K^{\mathrm{v}}_{\mathrm{RX}}>0$, we set

$$N^{\mathrm{v}}_{\sigma_{\mathrm{RX}}(1)}=N^{\mathrm{v}}_{\sigma_{\mathrm{RX}}(1)}-1,\quad \mathcal{S}^{\mathrm{NT}}_{\mathrm{RX}}=\{\mathcal{S}^{\mathrm{NT}}_{\mathrm{RX}},\sigma_{\mathrm{RX}}(1)\} \tag{43}$$

and we start at step $n+1$.

- If $K^{\mathrm{TX}}_{\mathrm{v}}=0$ and $K^{\mathrm{RX}}_{\mathrm{v}}=0$, the algorithm has reached its end.

Description of the Algorithm Each iteration step is divided into two procedures marked with the $1)$ and the $2)$, respectively.

The first process consists in finding all the generalized tightly-feasible sub-ICs by verifying condition (17). The tightly-feasible sub-ICs are obtained following similar steps as in the algorithm in Subsection III-B. It consists in the gradual increase of the set of TXs and the set of RXs so as to always obtain the ”most tight” sub-ICs. For each of these sets, equation (17) is tested to verify whether the setting is tightly-feasible. This is carried out in steps $1.c)$ and $1.d)$.

Once a tightly-feasible subset is found, the beamformers of this set are computed and the channels are replaced by the effective channels. This is done by the intermediate of the *virtual antenna configuration* which represents the antenna configuration of the effective channel obtained. This process corresponds to step $1.b)$.

At the end of procedure $1)$, the virtual antenna configuration obtained does not contain any tightly-feasible sub-ICs. This is critical for the second step because it means that reducing the number of variables by one *cannot* lead to the violation of (8). As a consequence, IA feasibility is preserved by ”removing” one antenna in procedure $2)$. The policy chosen in the algorithm consists in removing the antenna at the TX with the smallest number of antennas if there is at least one TX left [Cf. equation (42)] and otherwise at the RX with the smallest number of antennas [Cf. equation (43)].

Heuristics of the Algorithm Our algorithm is based on the insight gained previously that, in order to reduce the size of the CSIT needed, the iterative IA algorithm should be replaced by *successive ZF steps* between the TXs and the RXs. This is obtained by making artificially the antenna configuration more heterogeneous, i.e., concentrating the fulfillment of the IA constraints

at a limited number of RXs and TXs.

B. Toy-Example of the Incomplete CSIT-Algorithm in Super-Feasible Settings

Let us consider as a toy-example the homogeneous tightly-feasible $(2,2)^3$ IC and let us further assume that TX 3 and RX 2 have each one additional antenna so that the IC becomes $(2,2).(3,2).(2,3)$. We will now go through the steps of our CSIT allocation algorithm for non-tightly feasible ICs. Note that following Corollary 1, our algorithm brings no reduction for the tightly-feasible setting $(2,2)^3$.

- $n=1$: In the first part, the algorithm starts by verifying whether there is any tightly-feasible set. This is not the case here such that the second part begins and one antenna is removed at TX 1. The virtual IC obtained is then $(2,1).(3,2).(2,3)$.
- $n=2$: It is again verified whether there is any tightly-feasible set. Since TX 1 has only one antenna, it forms by itself a tightly-feasible set, so that its TX beamformer can be fixed and the antenna configuration is replaced by the virtual antenna configuration $(2,*).(2,2).(1,3)$. RX 3 has then only one antenna, so that we can obtain the virtual IC $(2,*).(2,1).(*,3)$. Once more, the same step applies to TX 2 to obtain $(1,*).(2,*).(*,3)$ and then again to RX 1 to get $(*,*).(2,*).(*,2)$. Finally, there is no tightly-feasible set so that procedure $1)$ ends and procedure $2)$ begins. Consequently, one antenna is removed at TX 3 to obtain the IC $(*,*).(2,*).(*,1)$.
- Step 3: TX 3 has one antenna left so that the IC $(*,*).(1,*).(*,*)$ is obtained. The same is done for RX 2 to obtain the IC $(*,*).(*,*).(*,*)$. Both the TX set and the RX set are empty so that the stopping criteria is reached and the algorithm returns the set containing the indices of the ”removed antennas” $\mathcal{S}^{\text{TX}}_{\text{NT}}=\{1,3\}$ and $\mathcal{S}^{\text{RX}}_{\text{NT}}=\emptyset$.

The CSIT allocation algorithm leads to ”remove” the antennas at TX 1 and TX 3 to obtain the IC $(2,1).(3,2).(2,2)$. This setting is tightly-feasible and we can run the CSIT allocation for tightly-feasible ICs described in Subsection III-B which returns the CSIT allocation

$$\mathcal{A}=\{\mathbf{A}^{(1)}=\mathbf{A}_{\emptyset,\emptyset},\mathbf{A}^{(2)}=\mathbf{A}_{\{3\},\{1,2\}},\mathbf{A}^{(3)}=\mathbf{A}_{\{1,3\},\{1,2,3\}}\}. \tag{44}$$

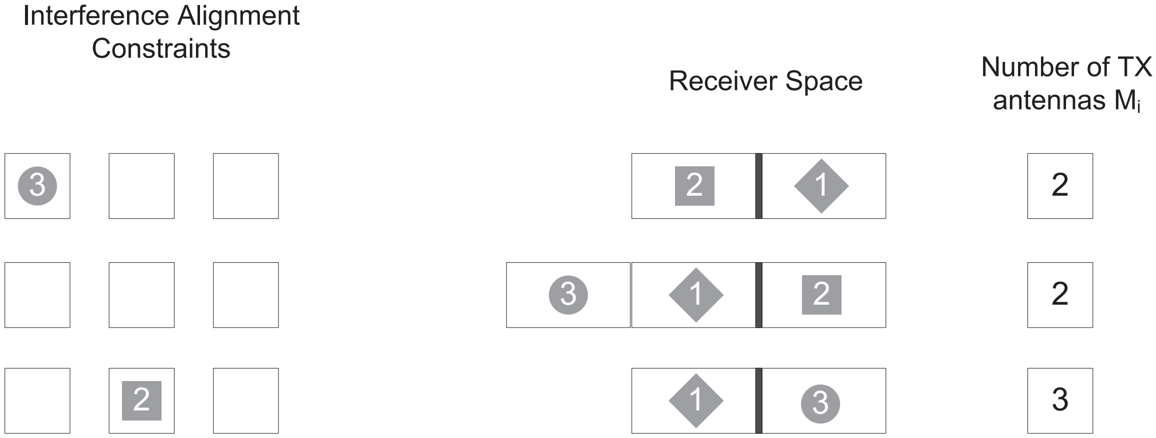

Fig. 2. Symbolic representation of the IA algorithm with incomplete CSIT for the super-feasible IC $(2,2).(3,2).(2,3)$.

The size of the CSIT allocation in (44) is equal to 20 while the complete CSIT allocation in the homogeneous setting $(2,2)^3$ has a size of 72. Thus, the additional antennas have been used to reduce the feedback size by more than a factor of 3. The IA algorithm for incomplete CSIT sharing which follows from this CSIT allocation is symbolically represented in Fig. 2.

V. Simulations

A. Tightly-Feasible Setting

We start by verifying by simulations that IA is indeed achieved by our new IA algorithm. We consider for the simulations the $(2,3).(2,4).(3,5).(3,2).(4,2)$ IC which has been studied in the example in Subsection III-D.

We show in Fig. 3 the average rate per user achieved in terms of the SNR. We compare the average rate achieved with our algorithm to the average rate obtained with the min-leakage IA algorithm based on full CSIT sharing. The proposed algorithm with incomplete CSIT achieves virtually the same performance as the min-leakage algorithm. Hence, the reduction of 60% of the feedback size (Cf. Subsection III-D) comes for ”free”, making it especially interesting in practice.

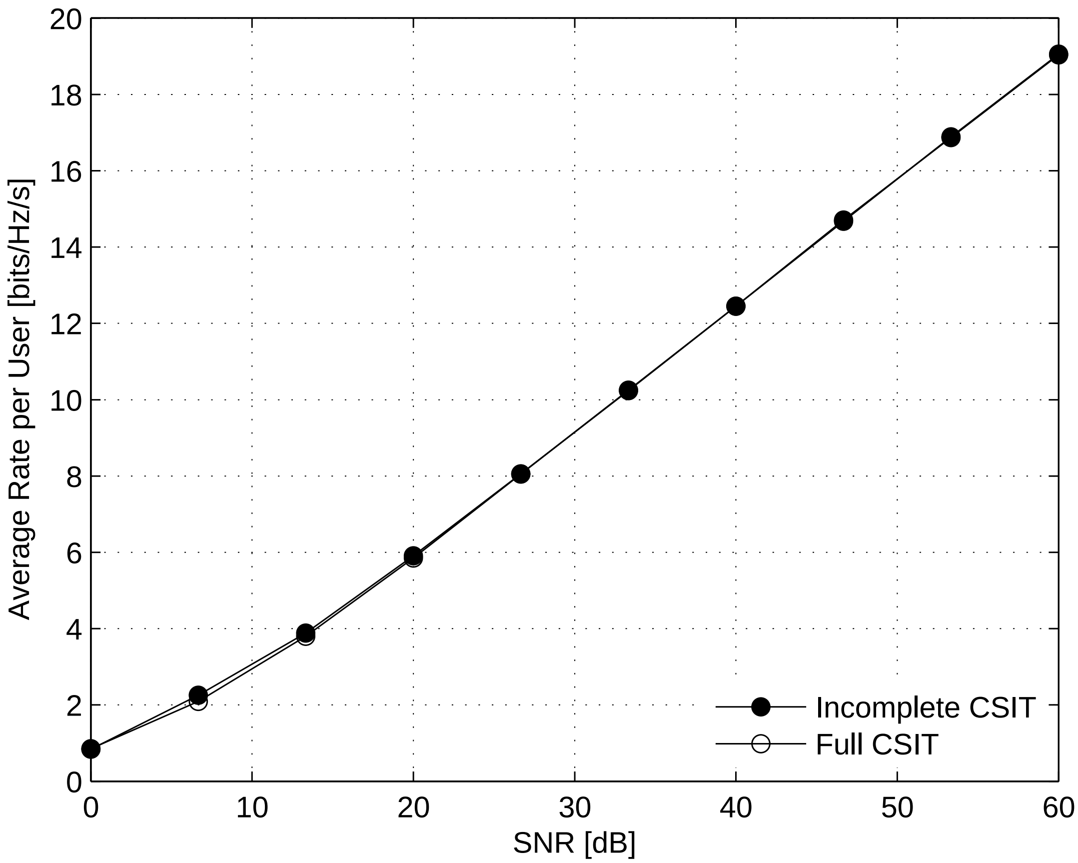

Fig. 3. Average rate per user in terms of the normalized TX power for the tightly-feasible IC $(2,3).(2,4).(3,5).(3,2).(4,2)$.

B. *Performance Evaluation of the CSIT allocation Algorithm*

We will now evaluate the feedback reduction obtained with our CSIT allocation policy in super-feasible settings. Since this gain depends on the antenna configuration, we average the performance over different antenna distributions across the TXs and the RXs. Hence, we show in Fig. 4 the size of the CSIT allocation for $K=3$ users when the antennas are allocated at random and uniformly to the TXs and the RXs.

We average over 1000 realizations and the proposed heuristic CSIT allocation policy is compared with the exhaustive search. The exhaustive search consists in testing all the possibilities

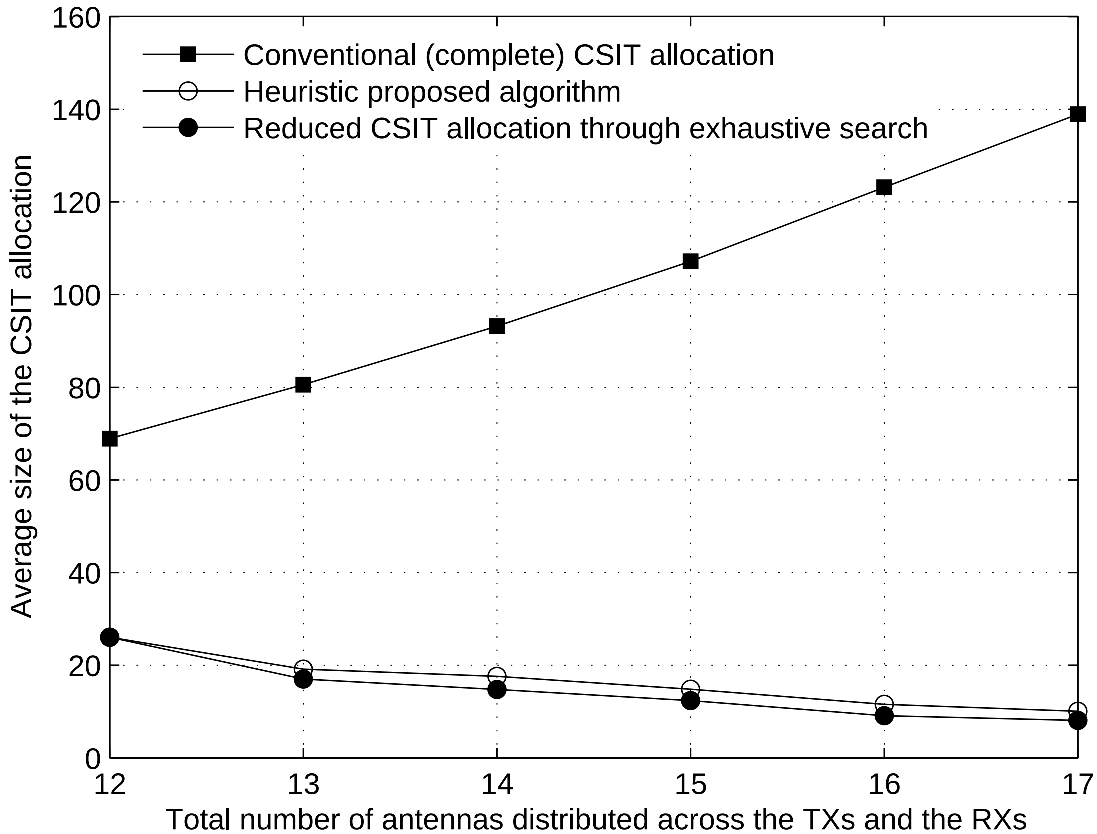

Fig. 4. Average CSIT allocation size in terms of the number of antennas distributed across the TXs and the RXs for $K = 3$ users.

for removing the additional antennas[5]. For reference, we also show the average size of the complete CSIT allocation. Only a small number of users is considered because of the exponential complexity of the exhaustive search.

If the aggregate number of antennas is strictly smaller than $K(K+1) = 12$, the feasibility condition (5) is not verified for the full IC such that IA is not feasible. If more than 12 antennas are available and the setting is feasible, each additional antenna is exploited by the heuristic algorithm to reduce the size of the CSIT allocation. This algorithm brings a reduction of the

[5]Note that a true exhaustive search through all the possible CSIT allocations is impossible even for trivial antenna configurations. Indeed, it requires testing for every TX the allocation of every channel coefficient whose number is much larger than the number of sub-ICs included in the full IC. For $M = N$, the total number of possibilities is $((K+1)/2)^2 K^2$, and for $K = 2$ and $M = N = 2$ it gives already 108 possibilities.

CSIT size which is only slightly smaller than the reduction brought by exhaustive search, but has a polynomial complexity.

VI. Discussion

IA feasibility is studied in the literature under the assumption of full CSIT sharing. In contrast, the relation between IA feasibility and CSIT allocation is investigated in this work. Specifically, it is shown how IA can be achieved in some cases without full CSIT sharing. When extra-antennas are available, the existence of a trade-off between the number of antennas available and the CSIT sharing requirements is shown.

Our approach brings a significant reduction of the feedback size while introducing no losses in terms of DoF compared to the conventional IA algorithm with full CSIT sharing.

Furthermore, IA with incomplete CSIT sharing raises additional interesting open problems that go beyond the scope of this paper. Firstly, proving the minimality of our reduced CSIT allocation (or finding the minimal CSIT allocation policy) could not be achieved due to the difficulty in deriving a lower bound for the minimal size of a CSIT allocation preserving IA feasibility. Another interesting problem is to extend the study to multiple streams transmissions. However, verifying IA feasibility represents already a difficult problem in this case so that the derivation of analytical results is expected to be challenging.

Finally, the analysis has been carried out by considering the DoF which models the performance at asymptotically high SNR. At low to medium SNR, aligning interference is possibly not optimal and it is expected that CSIT incompleteness will lead to some rate loss as beaforming capabilities are reduced. An interesting problem thus lies in the trade-off between CSIT sharing reduction and finite SNR rate performance. Such problems should be investigated in the future to translate the promising feedback reduction achieved in this work into savings in practical systems.

APPENDIX

A. Minimum Leakage Interference Algorithm

Many IA algorithms are already available in the literature [5]–[10] and each of them aims at maximizing the performance at finite SNR while converging to an IA solution at high SNR. The aim of this work is to study the feasibility of IA and not to improve on the performance of IA algorithm at finite SNR. Thus, we will use for the simulations the *minimum (min-) leakage algorithm* from [5]. It has the advantage of not requiring the knowledge of the direct channel but only the CSI required for fulfilling the IA constraints, i.e., the interfering channels.

The min-leakage algorithm can be described in our setting as follows. The algorithm minimizes the sum of the interference power created at the RXs which is called I_{IA} and is equal to

$$I_{\text{IA}} \triangleq \sum_{i=1}^{K} \sum_{k=1,k\neq i}^{K} |\boldsymbol{g}_i^{\text{H}} \mathbf{H}_{ik} \boldsymbol{t}_k|^2. \tag{45}$$

The algorithm is based on an alternating minimization in which the TX beamformers are first obtained from the RX beamformers as

$$\boldsymbol{t}_k = \text{eig}_{\min} \left(\sum_{i=1,i\neq k}^{K} \mathbf{H}_{ik}^{\text{H}} \boldsymbol{g}_i \boldsymbol{g}_i^{\text{H}} \mathbf{H}_{ik} \right). \tag{46}$$

Similarly, the RX beamformers at all RXs are then obtained from the TX beamformers as

$$\boldsymbol{g}_k = \text{eig}_{\min} \left(\sum_{i=1,i\neq k}^{K} \mathbf{H}_{ki} \boldsymbol{t}_i \boldsymbol{t}_i^{\text{H}} \mathbf{H}_{ki}^{\text{H}} \right). \tag{47}$$

The TX and RX beamformers are updated iteratively until convergence to a local minimizer.

B. Proof of Theorem 2

For $\mathcal{I} \subseteq \mathcal{J} = \{(i,j) | 1 \leq i,j \leq K, i \neq j\}$, we define the sets

$$\mathcal{S}_{\text{TX}}(\mathcal{I}) \triangleq \{j | \exists k', (k',j) \in \mathcal{I}\}, \quad \mathcal{S}_{\text{RX}}(\mathcal{I}) \triangleq \{k | \exists j', (k,j') \in \mathcal{I}\}. \tag{48}$$

Hence, $\mathcal{S}_{\text{RX}}(\mathcal{I})$ and $\mathcal{S}_{\text{TX}}(\mathcal{I})$ contain respectively the set of RXs and the set of TXs appearing in at least one equation of the set of equations $\mathcal{I}$. With these notations, equation (5) can be

rewritten as

$$\forall \mathcal{I} \subseteq \mathcal{J}, \qquad |\mathcal{I}| \leq \sum_{k \in \mathcal{S}_{\text{TX}}(\mathcal{I})} (M_k - 1) + \sum_{j \in \mathcal{S}_{\text{RX}}(\mathcal{I})} (N_j - 1). \tag{49}$$

Adding equations to $\mathcal{I}$ without increasing $\mathcal{S}_{\text{RX}}(\mathcal{I})$ or $\mathcal{S}_{\text{TX}}(\mathcal{I})$ makes condition (49) tighter. Hence, it is only necessary to verify (49) for the sets of equations made of all the equations generated by the RXs in $\mathcal{S}_{\text{RX}}(\mathcal{I})$ and the TXs in $\mathcal{S}_{\text{TX}}(\mathcal{I})$.

C. Proof of Theorem 3

Proof: We have by assumption that $\mathcal{N}_{\text{var}}(\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}}) = \mathcal{N}_{\text{eq}}(\mathcal{S}_{\text{RX}}, \mathcal{S}_{\text{TX}})$. If any additional IA constraint is considered, the number of equations becomes then larger than the number of equations and IA cannot be achieved. Hence, setting the TX beamformers and the RX beamformers contained in this sub-IC to fulfill the IA constraints solely inside this IC represents in fact a *necessary* condition for achieving IA in the full IC. As a consequence, IA feasibility is preserved by letting the TXs in the tightly-feasible sub-IC have only the knowledge of the CSI relative to this sub-IC. ■

D. Proof of Theorem 4

Proof: Let us consider wlog the precoding at TX j. By construction, TX j is allocated with the CSI relative to the IC formed by the pair of sets $(\mathcal{S}_{\text{RX}}^{(j)}, \mathcal{S}_{\text{TX}}^{(j)})$, which is *tightly-feasible*. A side result of the proof of Theorem 3 is that setting the beamformers in a tightly-feasible subset of TXs and RXs to align interference in this sub-IC, does not reduce the feasibility of IA in the full IC. Thus, if all the TXs included in $\mathcal{S}_{\text{TX}}^{(j)}$ would design jointly their beamformers with the other TXs adapting to these TX beamformers, IA feasibility would then be preserved. Yet, all the TXs in $\mathcal{S}_{\text{TX}}^{(j)}$ do not necessarily share the same CSIT and thereby cannot necessarily design jointly the beamformers. Thus, it remains to prove that all the TXs included in $\mathcal{S}_{\text{TX}}^{(j)}$ design their beamformers in such a way that IA is achieved inside this sub-IC.

By inspection of the CSIT allocation algorithm, the CSIT allocations of all the TXs contained in $\mathcal{S}_{\text{TX}}^{(j)}$ are included in the CSIT of TX j. Thus, TX j can compute the beamformers of these

TXs following the IA algorithm for incomplete CSIT exactly as it is done at these TXs. This ensures the coherency between the beamformers of all the TXs in $\mathcal{S}_{\text{TX}}^{(j)}$ so that IA is achieved. ■

REFERENCES

[1] P. de Kerret and D. Gesbert, "MIMO Interference Alignment Algorithms With Hierarchical CSIT," in *Proc. ISWCS 2012*.

[2] M. A. Maddah-Ali, A. S. Motahari, and A. K. Khandani, "Communication Over MIMO X Channels: Interference Alignment, Decomposition, and Performance Analysis," *IEEE Transactions on Information Theory*, vol. 54, no. 8, pp. 3457–3470, August 2008.

[3] V. R. Cadambe and S. A. Jafar, "Interference Alignment and Degrees of Freedom of the K-User Interference Channel," *IEEE Transactions on Information Theory*, vol. 54, no. 8, pp. 3425–3441, August 2008.

[4] D. Gesbert, S. Hanly, H. Huang, S. Shamai (Shitz), O. Simeone, and W. Yu, "Multi-Cell MIMO Cooperative Networks: A New Look at Interference," *IEEE Journal on Selected Areas in Communications*, vol. 28, no. 9, pp. 1380–1408, December 2010.

[5] K. Gomadam, V. Cadambe, and S. A. Jafar, "Approaching the Capacity of Wireless Networks through Distributed Interference Alignment," in *Proc. GLOBECOM 2008*.

[6] D. Schmidt, C. Shi, R. Berry, M. Honig, and W. Utschick, "Minimum Mean Squared Error Interference Alignment," in *Proc. Asilomar 2009*.

[7] K. R. Kumar, "An Iterative Algorithm for Joint Signal and Interference Alignment," in *Proc. ISIT 2010*.

[8] I. Santamaría, Ó. González, R. W. Heath, and S. W. Peters, "Maximum Sum-Rate Interference Alignment Algorithms for MIMO Channels," in *Proc. GLOBECOM 2010*.

[9] D. Papailiopoulos and A. Dimakis, "Interference Alignment as a Rank Constrained Rank Minimization," in *Proc. GLOBECOM 2010*.

[10] S. W. Peters and R. W. Heath, "User Partitioning for Less Overhead in MIMO Interference Channels," *IEEE Transactions on Wireless Communications*, vol. 11, no. 2, pp. 592–603, February 2012.

[11] J. A. García-Naya, L. Castedo, Ó. González, D. Ramírez, and I. Santamaría, "Experimental Evaluation of Interference Alignment Under Imperfect Channel State Information," in *Proc. EUSIPCO 2011*.

[12] R. Tresch and M. Guillaud, "Cellular Interference Alignment with Imperfect Channel Knowledge," in *Proc. ICC 2009*.

[13] J. Thukral and H. Boelcskei, "Interference Alignment with Limited Feedback," in *Proc. ISIT 2009*.

[14] R. T. Krishnamachari and M. K. Varanasi, "Interference Alignment under Limited Feedback for MIMO Interference Channels," in *Proc. ISIT 2010*.

[15] C. Shi, R. Berry, and M. Honig, "Adaptive Beamforming in Interference Networks via Bi-Directional Training," in *Proc. CISS 2010*.

[16] O. E. Ayach and R. W. Heath, "Interference Alignment with Analog Channel State Feedback," *IEEE Transactions on Wireless Communications*, vol. 11, no. 2, pp. 626–636, February 2012.

[17] S. W. P. O. El Ayach and R. W. Heath., "The Practical Challenges of Interference Alignment," to appear in IEEE Wireless Communications Magazine. Available under arXiv: http://arxiv.org/pdf/1206.4755v1.pdf.

[18] J. H. Lee, W. Choi, and D. J. Love, "Achievable DoF and Its User Scaling Law for Opportunistic User Selection in a K-transmitter SIMO Interference Channel," submitted to IEEE Transactions on Information Theory, 2011. Available under arXiv: http://arxiv.org/abs/1111.1051.

[19] S. Cho, H. Chae, K. Huang, D. Kim, V. K. N. Lau, H. Seo, and B. Kim, "Feedback-Topology Designs for Interference Alignment in MIMO Interference Channels," submitted to IEEE Transactions on Signal Processing, May 2011. Available under arXiv: http://arxiv.org/pdf/1105.5476v4.pdf.

[20] S. Cho, H. Chae, K. Huang, D. Kim, V. K. N. Lau, and H. Seo, "Efficient Feedback Design for Interference Alignment in MIMO Interference Channel," in *Proc. VTC Spring 2012*.

[21] C. Suh and D. N. C. Tse, "Interference Alignment for Cellular Networks," in *Proc. Allerton 2008*.

[22] C. Suh, M. Ho, and D. N. C. Tse, "Downlink Interference Alignment," *IEEE Transactions on Communications*, vol. 59, no. 9, pp. 2616–2626, September 2011.

[23] C. Yetis, T. Gou, S. A. Jafar, and A. H. Kayran, "On Feasibility of Interference Alignment in MIMO Interference Networks," *IEEE Transactions on Signal Processing*, vol. 58, no. 9, pp. 4771–4782, September 2010.

[24] G. Bresler, D. Cartwright, and D. N. C. Tse, "Geometry of the 3-User MIMO Interference Channel," in *Proc. Allerton 2011*.

[25] G. Bresler, D. Cartwright, and D. Tse, "Feasibility of interference alignment for the MIMO interference channel: The symmetric square case," in *Proc. ITW 2011*.

[26] M. Razaviyayn, G. Lyubeznik, and Z.-Q. Luo, "On the Degrees of Freedom Achievable Through Interference Alignment in a MIMO Interference Channel," *IEEE Transactions on Signal Processing*, vol. 60, no. 2, pp. 812–821, February 2012.

[27] O. Gonzalez, C. Beltran, and I. Santamaría, "On the Feasibility of Interference Alignment for the K-User MIMO Channel with Constant Coefficients," submitted to IEEE Transactions on Information Theory, February 2012. Available under Arxiv: http://arxiv.org/abs/1202.0186.